Line, Surface, and Volume Integrals of Vector Fields

Mathematical Physics

Semester–IV, MJC–5

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1. Introduction

In Mathematical Physics, many physical quantities such as **work done by force**, **flux of electric and magnetic fields**, **fluid flow**, and **distribution of mass or charge** are described using **vector integrals**. Depending on whether the integration is carried out over a **curve**, **surface**, or **volume**, vector integrals are classified into:

1. Line integral
2. Surface integral
3. Volume integral

2. Line Integral of a Vector Field

Definition

A **line integral of a vector field** is defined as the integral of a vector field taken **along a given curve**.

Let

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

The line integral is:

$$\int_C \vec{F} \cdot d\vec{r}$$

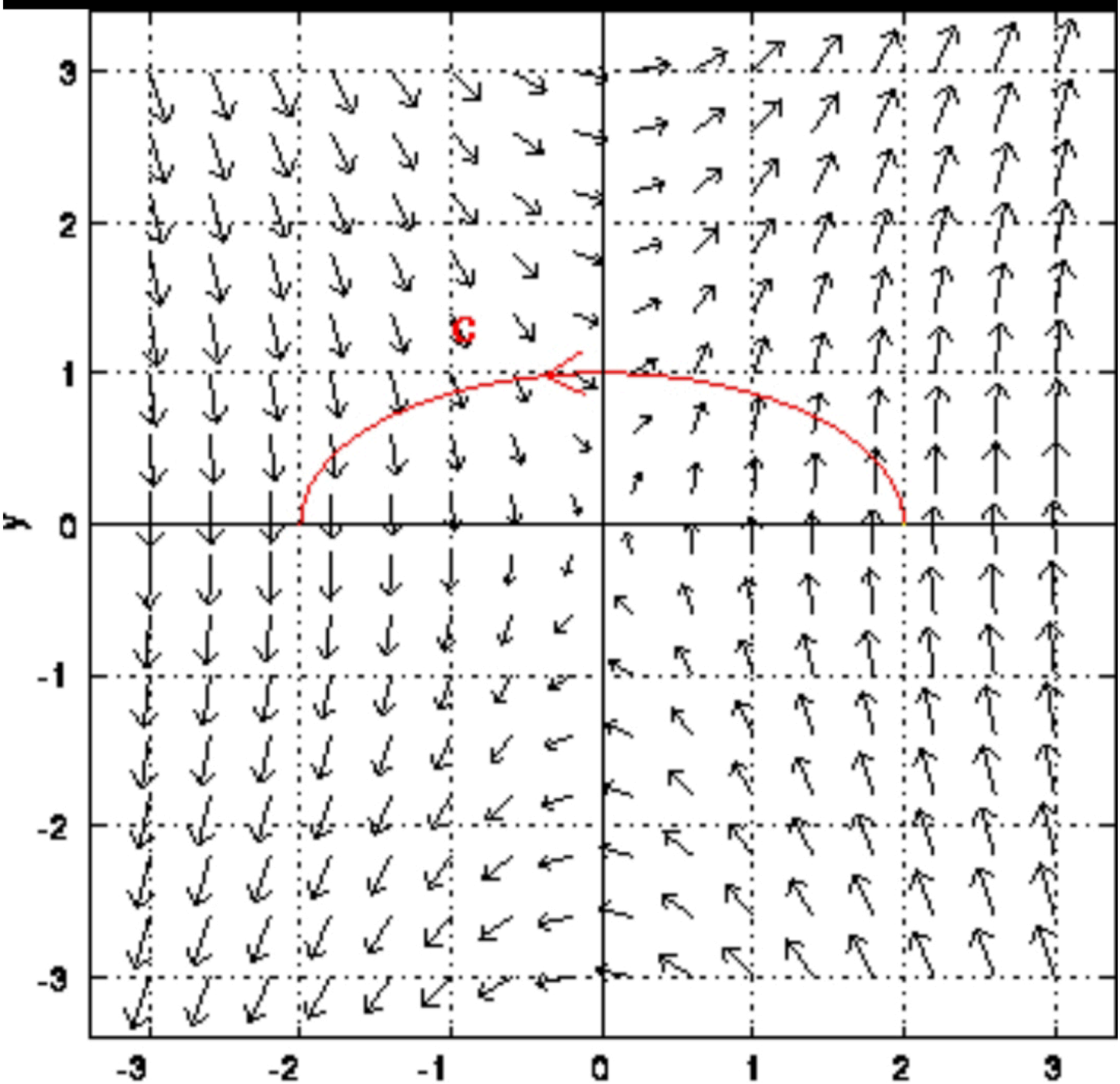
where

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

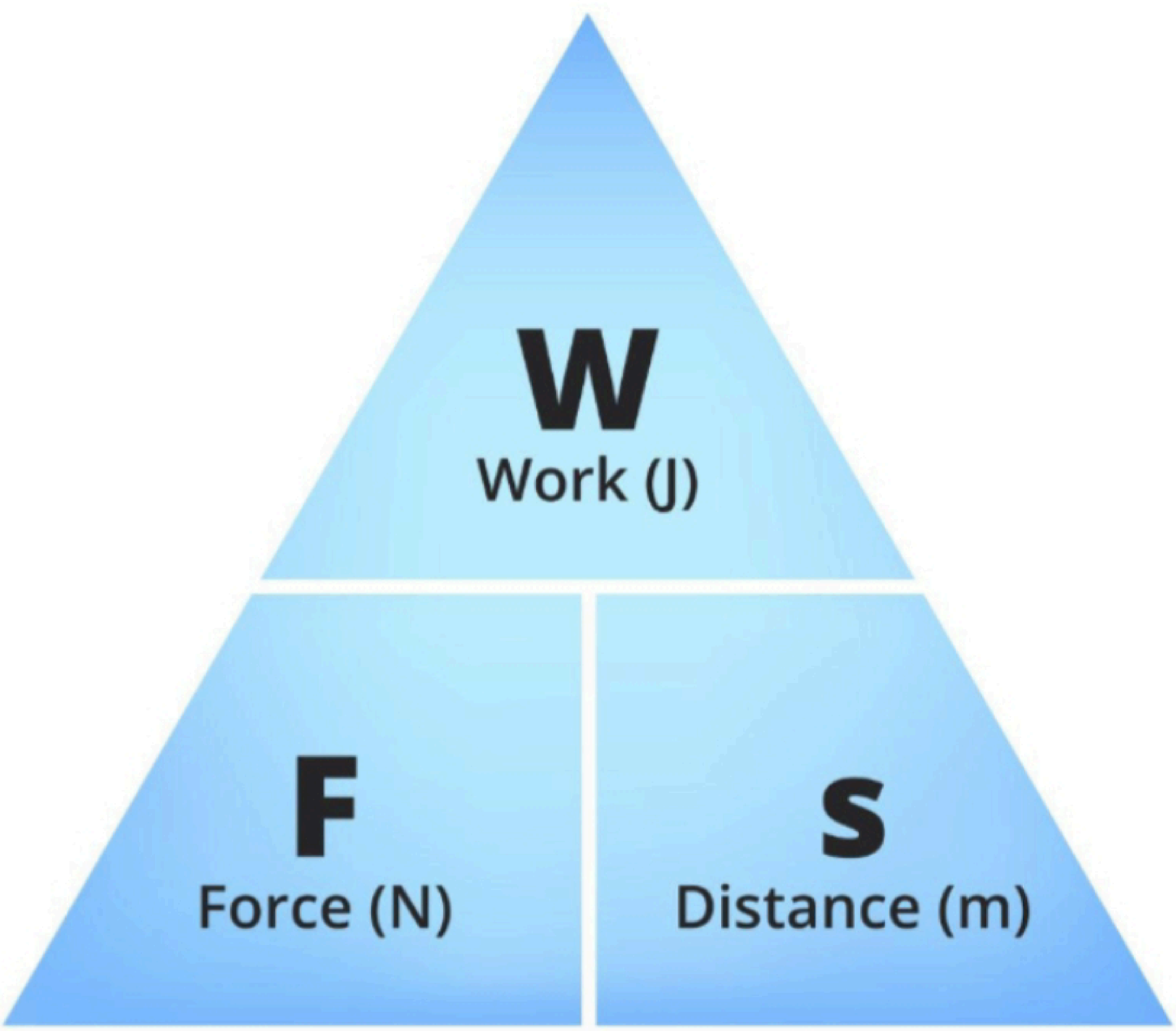
Hence,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (F_x dx + F_y dy + F_z dz)$$

Line Integrals of Vector Fields



WORK FORMULA



$$W = F \cdot s$$

$$F = W : s$$

$$s = W : F$$

Physical Significance

- Represents the **work done** by a force field in moving a particle along a path.
- In general, it **depends on the path** taken.

Conservative Field

If the vector field is conservative:

$$\vec{F} = \nabla\phi$$

then the line integral is **path independent**:

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

3. Surface Integral of a Vector Field

Definition

The **surface integral** of a vector field represents the **flux of the vector field through a given surface**.

If a vector field $\vec{F}$ passes through a surface **S**, then the surface integral is:

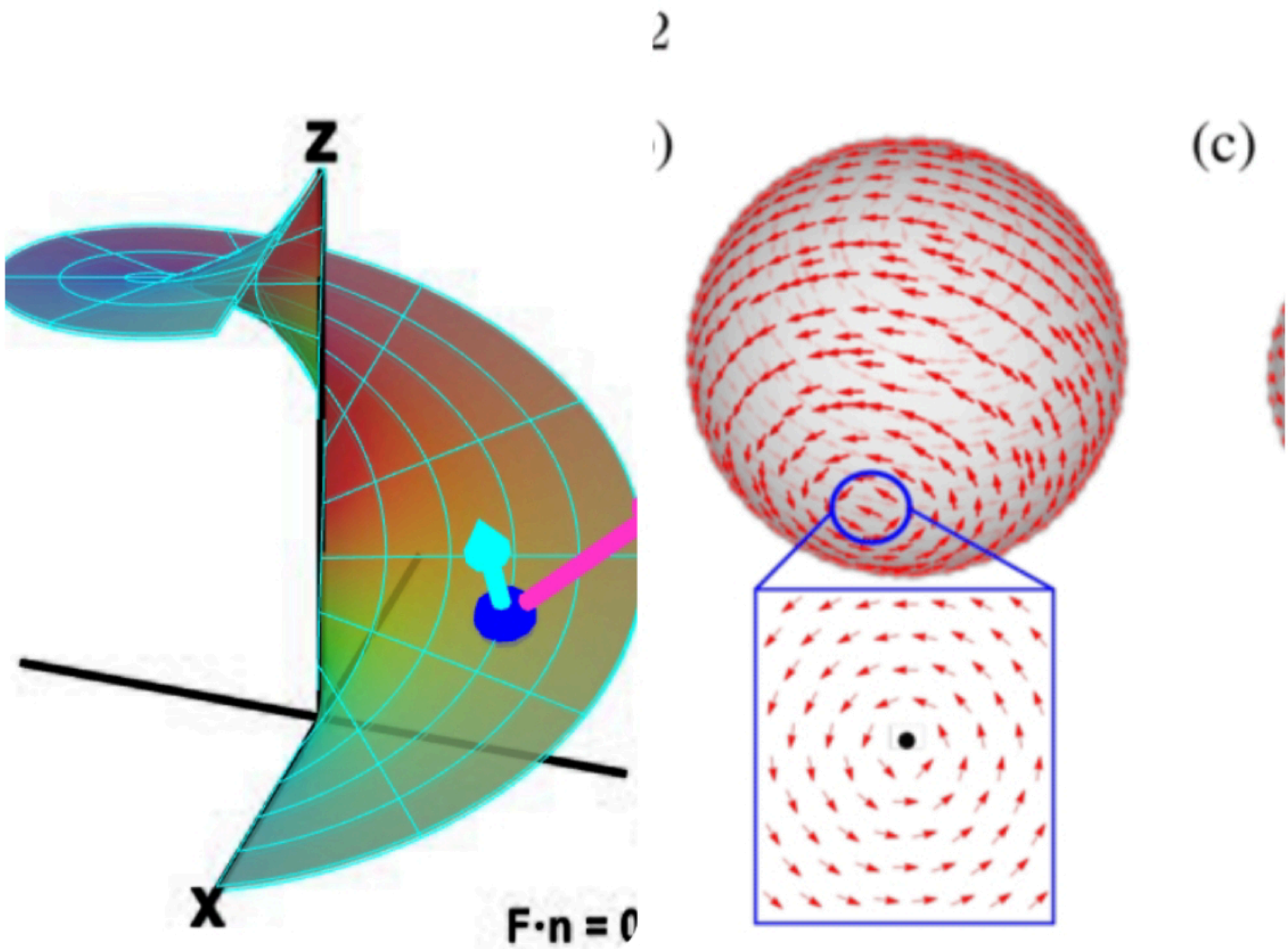
$$\iint_S \vec{F} \cdot d\vec{S}$$

where

$$d\vec{S} = \hat{n}\, dS$$

Mathematical Expression

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (F_x n_x + F_y n_y + F_z n_z)\, dS$$



Physical Interpretation

- Measures **flux** of electric field, magnetic field, or fluid velocity.
- Widely used in **el** **magnetism** and **fluid mechanics**.

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4. Volume Integral of a Vector Field

Definition

A **volume integral** is used to integrate a scalar or vector quantity over a **three-dimensional region**.

For a scalar function $\phi(x, y, z)$:

$$\iiint_V \phi \, dV$$

For a vector field $\vec{F}$:

$$\iiint_V \vec{F} \, dV$$

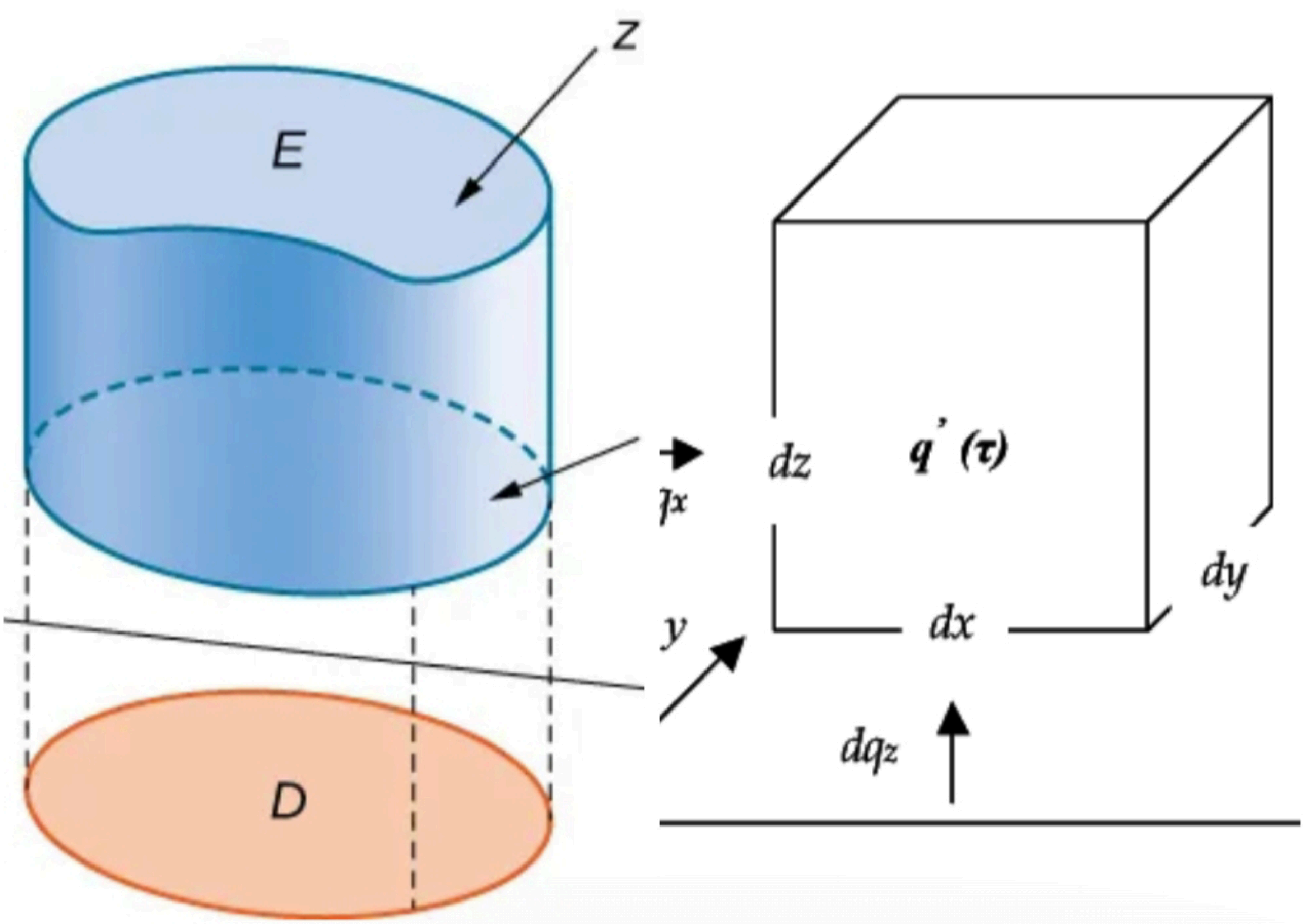
Cartesian Form

In Cartesian coordinates:

$$dV = dx\,dy\,dz$$

Thus,

$$\iiint_V \vec{F}\,dV = \iiint_V (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\,dx\,dy\,dz$$



Applications

- Calculation of **total mass**, **charge**, or **energy** inside a region
- Used in **Gauss's law**, **continuity equation**, and **field theory**

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5. Important Theorems of Vector Integration

(a) Gradient Theorem

$$\int_C \nabla\phi \cdot d\vec{r} = \phi(B) - \phi(A)$$

(b) Stokes' Theorem

It relates a **line integral** to a **surface integral**:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

(c) Gauss Divergence Theorem

It relates a **surface integral** to a **volume integral**:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F})\, dV$$

6. Comparison of Vector Integrals

Integral Type	Region	Physical Meani
Line Integral	Curve	Work done
Surface Integral	Surface	Flux
Volume Integral	Volume	Total quantity

7. Conclusion

Line, surface, and volume integrals of vector fields are fundamental tools of **Mathematical Physics**. They provide a mathematical description of physical laws in **mechanics, electromagnetism, and fluid dynamics**. The vector integral theorems establish deep connections between different forms of integration and simplify the analysis of physical systems.